- If a periodic signal x is *continuous* and its Fourier series coefficients c_k are *absolutely summable* (i.e $\sum_{k=-\infty}^{\infty} |c_k| < \infty$), then the Fourier series representation of x converges *uniformly* (i.e., pointwise at the same rate everywhere.(
- Since, in practice, we often encounter signals with discontinuities (e.g., a square wave), the above result is of somewhat limited value.

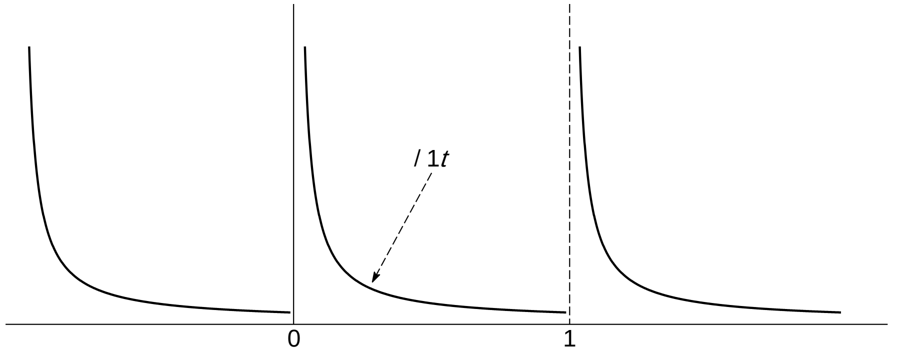
- If a periodic signal x has *finite energy* in a single period (i.e., $\int_T |x(t)|^2 dt < \infty$), the Fourier series converges in the *MSE* sense.
- Since, in situations of practice interest, the finite-energy condition in the above theorem is typically satisfied, the theorem is usually applicable.
- It is important to note, however, that MSE convergence (i.e., $E = 0$) does not necessarily imply pointwise convergence (i.e., $\tilde{x}(t) = x(t)$ for all t.(
- Thus, the above convergence theorem does not provide much useful information regarding the value of $\tilde{x}(t)$ at specific values of t.
- Consequently, the above theorem is typically most useful for simply determining if the Fourier series converges.

- The Dirichlet conditions for the periodic signal x are as follows:
  1. Over a single period, x is *absolutely integrable* (i.e., $\int_T |x(t)|\,dt < \infty$). Over a
  2. single period, x has a finite number of maxima and minima (i.e., x is of *bounded variation*).
  3. Over any finite interval, x has a *finite number of discontinuities*, each of which is *finite*.
- If a periodic signal x satisfies the *Dirichlet conditions*, then:
  1. The Fourier series converges pointwise everywhere to x, except at the points of discontinuity of x.
  2. At each point $t = t_a$ of discontinuity of x, the Fourier series $\tilde{x}$ converges to

$$\tilde{x}(t_a) = \tfrac{1}{2}\left[x(t_a^-) + x(t_a^+)\right]$$

where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the signal x on the left- and right-hand sides of the discontinuity, respectively.

- Since most signals tend to satisfy the Dirichlet conditions and the above convergence result specifies the value of the Fourier series at every point, this result is often very useful in practice.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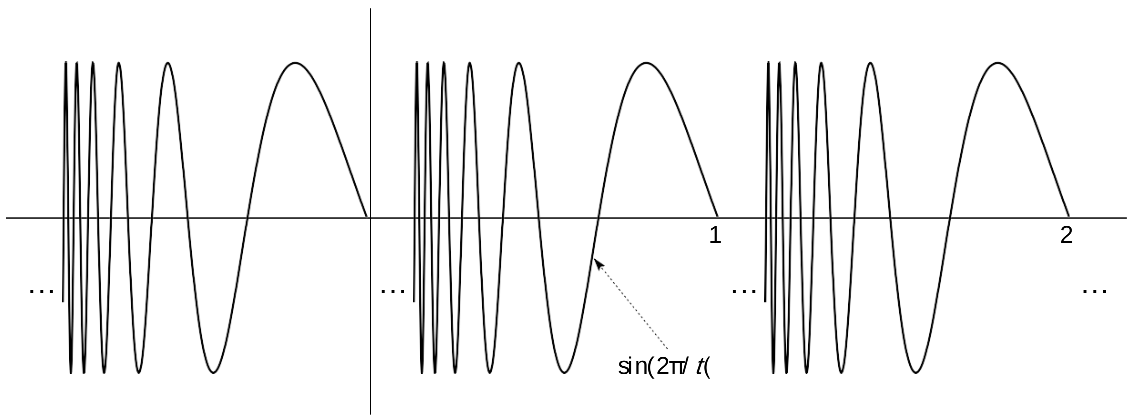
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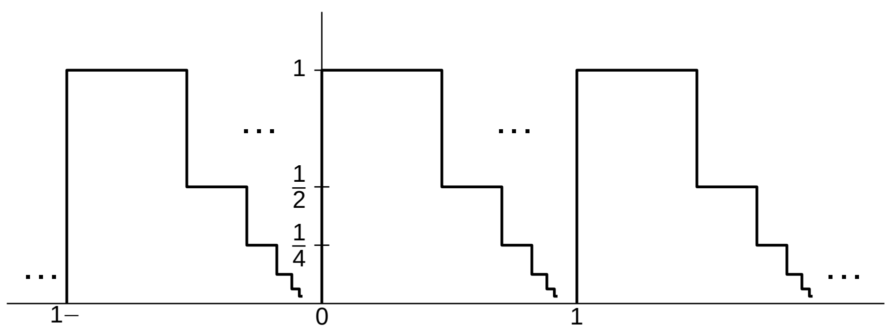
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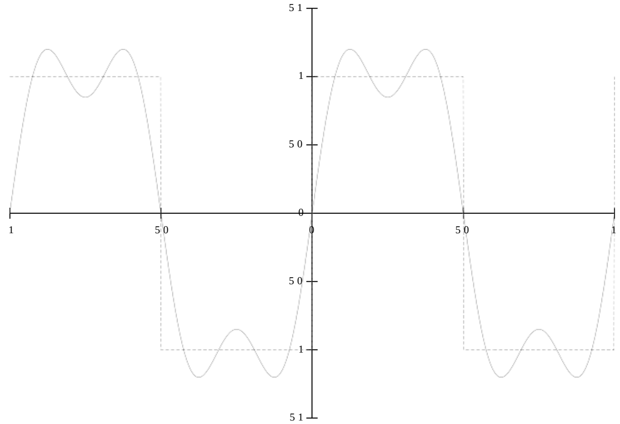
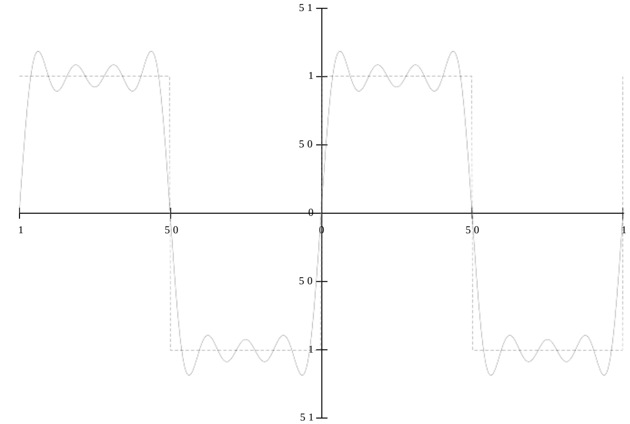
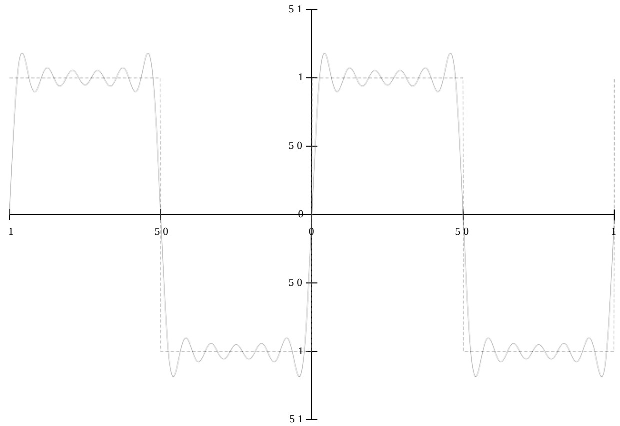
- In practice, we frequently encounter signals with discontinuities.
- When a signal x has discontinuities, the Fourier series representation of x does not converge uniformly (i.e., at the same rate everywhere.(
- The rate of convergence is much slower at points in the vicinity of a discontinuity.
- Furthermore, in the vicinity of a discontinuity, the truncated Fourier series x_N exhibits ripples, where the peak amplitude of the ripples does not seem to decrease with increasing N.
- As it turns out, as N increases, the ripples get compressed towards discontinuity, but, for any finite N, the peak amplitude of the ripples remains approximately constant.
- This behavior is known as Gibbs phenomenon.
- The above behavior is one of the weaknesses of Fourier series (i.e., Fourier series converge very slowly near discontinuities.(



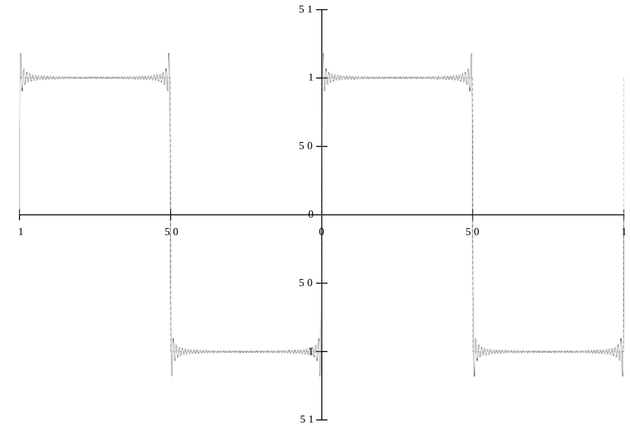
Fourier series truncated after the 3rd harmonic components



Fourier series truncated after the 7th harmonic components



Fourier series truncated after the 11th harmonic components



Fourier series truncated after the 101th harmonic components

Section 4.3

Properties of Fourier Series

$$x(t) \xleftrightarrow{\text{CTFS}} a_k \quad \text{and} \quad y(t) \xleftrightarrow{\text{CTFS}} b_k$$

Property	Time Domain	Fourier Domain
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha a_k + \beta b_k$
Translation	$x(t - t_0)$	$e^{-jk(2\pi/T)t_0} a_k$
Reflection	$x(-t)$	a_{-k}
Conjugation	$x^*(t)$	a^*_{-k}
Even symmetry	x even	a even
Odd symmetry	x odd	a odd
Real	$x(t)$ real	$a_k = a^*_{-k}$

Property	
Parseval's relation	$\frac{1}{T}\int_T \lvert x(t)\rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert a_k\rvert^2$

- Let x and y be two periodic signals with the same period. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k$ and $y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k$, then

$$\alpha x(t) + \beta y(t) \overset{\text{CTFS}}{\longleftrightarrow} \alpha a_k + \beta b_k,$$

where α and β are complex constants.

- That is, a linear combination of signals produces the same linear combination of their Fourier series coefficients.

- Let x denote a periodic signal with period T and the corresponding frequency $\omega_0 = 2\pi/T$. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$, then

$$x(t-t_0) \overset{\text{CTFS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} c_k = e^{-jk(2\pi/T)t_0} c_k,$$

  where t_0 is a real constant.

- In other words, time shifting a periodic signal changes the argument (but not magnitude) of its Fourier series coefficients.

- Let x denote a periodic signal with period T and the corresponding frequency $\omega_0 = 2\pi/T$. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$, then

$$x(-t) \overset{\text{CTFS}}{\longleftrightarrow} c_{-k}.$$

- That is, time reversal of a signal results in a time reversal of its Fourier series coefficients.

- For a T-periodic function x with Fourier series coefficient sequence c, the following properties hold:

$$x^*(t) \overset{\text{CTFS}}{\longleftrightarrow} c^*_{-k}$$

- In other words, conjugating a signal has the effect of time reversing and conjugating the Fourier series coefficient sequence.

- For a T-periodic function x with Fourier series coefficient sequence c, the following properties hold:

$$x \text{ is even} \Leftrightarrow c \text{ is even}; \quad \text{and}$$
$$x \text{ is odd} \Leftrightarrow c \text{ is odd}.$$

- In other words, the even/odd symmetry properties of x and c always match.

- A signal x is *real* if and only if its Fourier series coefficient sequence c satisfies

$$c_k = c^*_{-k} \text{ for all } k$$

(i.e., c has *conjugate symmetry*).
- Thus, for a real-valued signal, the negative-indexed Fourier series coefficients are *redundant*, as they are completely determined by the nonnegative-indexed coefficients.
- From properties of complex numbers, one can show that $c_k = c^*_{-k}$ is equivalent to

$$|c_k| = |c_{-k}| \quad \text{and} \quad \arg c_k = -\arg c_{-k}$$

)i.e., $|c_k|$ is *even* and $\arg c_k$ is *odd*.(
- Note that x being real does *not* necessarily imply that c is real.

- For a T-periodic function x with Fourier-series coefficient sequence c, the following properties hold:
  1. c_0 is the average value of x over a single period;
  2. x is real and even $\Leftrightarrow$ c is real and even; and
  3. x is real and odd $\Leftrightarrow$ c is purely imaginary and odd.